MA 580; Introduction to Linear Equations

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Study Chapter 3 of the pink book an Chapter 4 of the notes now!

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Part III: Linear Equations, Introduction

Linear Systems of Equations

- $\mathbf{Ax} = \mathbf{b}$
  - Solution $\mathbf{x}^* = \mathbf{A}^{-1} b$
  - Error in $\mathbf{x}$ (LA notation) $e = \mathbf{x}^* - \mathbf{x}$
  - Error in $\mathbf{x}$ (NL notation) $e = \mathbf{x} - \mathbf{x}^*$ (think Taylor)

Upper and Lower Bounds for $\|\mathbf{Ax}\|$

The definition of induced norm tells us that

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\|\|\mathbf{x}\|,$$

and

$$\|\mathbf{x}\| = \|\mathbf{A}^{-1}\mathbf{Ax}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{Ax}\|.$$

This means that

$$\|\mathbf{A}^{-1}\|^{-1}\|\mathbf{x}\| \leq \|\mathbf{Ax}\|.$$

Conditioning

The condition number of $\mathbf{A}$ is

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$$

If $\kappa(\mathbf{A})$ is large, you are not likely to solve $\mathbf{Ax} = \mathbf{b}$ very well.

Does Checking your Answer mean anything?

Suppose your goal is to reduce the relative error

$$\frac{\|\mathbf{e}\|}{\|\mathbf{x}^*\|}.$$

Is a reduction in the relative residual

$$\frac{\|\mathbf{b} - \mathbf{Ax}\|}{\|\mathbf{b}\|}$$

a useful surrogate for a reduction in the relative error?

The Check-your-Answer Theorem

Temporarily we will use $\mathbf{e} = \mathbf{x}^* - \mathbf{x}$ so that

$$\mathbf{r} \equiv \mathbf{b} - \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{e}.$$

Theorem: If $\mathbf{A}$ is nonsingular then

$$\kappa(\mathbf{A})^{-1} \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \le \frac{\|\mathbf{e}\|}{\|\mathbf{x}^*\|} \le \kappa(\mathbf{A}) \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

Talk about what this means.

Proof: 1

Note that $\mathbf{r} = \mathbf{A}\mathbf{e}$ implies that so

$$\|\mathbf{e}\| = \|\mathbf{A}^{-1}\mathbf{A}\mathbf{e}\| \le \|\mathbf{A}^{-1}\|\|\mathbf{A}\mathbf{e}\| = \|\mathbf{A}^{-1}\|\|\mathbf{r}\|$$

and

$$\|\mathbf{r}\| = \|\mathbf{A}\mathbf{e}\| \le \|\mathbf{A}\|\|\mathbf{e}\|.$$

Proof: 2

Use the upper and lower bounds for $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ to get

$$\frac{\|\mathbf{e}\|}{\|\mathbf{x}^*\|} \leq \frac{\|\mathbf{A}^{-1}\|\|\mathbf{r}\|}{\|\mathbf{A}\|^{-1}\|\mathbf{b}\|} = \kappa(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

Combine with the first part and

$$\frac{\|\mathbf{e}\|}{\|\mathbf{x}^*\|} \geq \frac{\|\mathbf{A}\|^{-1}\|\mathbf{r}\|}{\|\mathbf{A}^{-1}\|\|\mathbf{b}\|} = \kappa(\mathbf{A})^{-1}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

as asserted.

Forward and Backward Error

Suppose $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ and you compute $\mathbf{x} \approx \mathbf{x}^*$.
The **forward error** is $\|\mathbf{x}^* - \mathbf{x}\|$.
Now, suppose you can show (and in many cases you can) that $\mathbf{x}$ is the exact solution to a nearby system

$$(\mathbf{A} + \mathbf{E})\mathbf{x} = \mathbf{b} + \delta\mathbf{b}$$

The pair $\mathbf{E}$ and $\delta\mathbf{b}$ is the **backward error**.

Connection between forward and backward errors

Theorem: Suppose $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, $(\mathbf{A} + \mathbf{E})\mathbf{x} = \mathbf{b} + \delta\mathbf{b}$, and

$$\|\mathbf{E}\| \le \frac{1}{2\|\mathbf{A}^{-1}\|}$$

$$\frac{\|\mathbf{x}^* - \mathbf{x}\|}{\|\mathbf{x}^*\|} \le 2\kappa(\mathbf{A}) \left(\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right).$$

Lemma

$\|\mathbf{E}\| \le \frac{1}{2\|\mathbf{A}^{-1}\|}$ implies that $\mathbf{A}+\mathbf{E}$ is nonsingular and

$$\|(\mathbf{A}+\mathbf{E})^{-1}\| \le 2\|\mathbf{A}^{-1}\|.$$

Proof of lemma: For any $\mathbf{u}$

$$\begin{aligned} \|(\mathbf{A}+\mathbf{E})\mathbf{u}\| &= \|(I+\mathbf{E}\mathbf{A}^{-1})\mathbf{A}\mathbf{u}\| \ge (1-\|\mathbf{E}\|\|\mathbf{A}^{-1}\|)\|\mathbf{A}\mathbf{u}\| \\ &\ge \|\mathbf{A}^{-1}\|^{-1}\|\mathbf{u}\|/2 \end{aligned}$$

and that's it.

Proof: 1

Subtract

$$(\mathbf{A} + \mathbf{E})\mathbf{x}^* = \mathbf{b} + \mathbf{E}\mathbf{x}^*$$

from $(\mathbf{A} + \mathbf{E})\mathbf{x} = \mathbf{b} + \delta\mathbf{b}$ to get

$$(\mathbf{A} + \mathbf{E})(\mathbf{x} - \mathbf{x}^*) = \delta\mathbf{b} - \mathbf{E}\mathbf{x}^*$$

So . . .

Proof: 2

the lemma says that $\mathbf{A} + \mathbf{E}$ is nonsingular and

$$\mathbf{x} - \mathbf{x}^* = (\mathbf{A} + \mathbf{E})^{-1}(\delta\mathbf{b} - \mathbf{E}\mathbf{x}^*)$$

which implies that

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}^*\| \le \quad & 2\|\mathbf{A}^{-1}\|(\|\mathbf{E}\|\|\mathbf{x}^*\| + \|\delta\mathbf{b}\|) \\ & = 2\kappa(\mathbf{A})\|\mathbf{x}^*\| \left(\frac{\|\mathbf{E}\|}{\|\mathbf{A}\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{A}\|\|\mathbf{x}^*\|} \right). \end{aligned}$$

Proof: 3

We're there because

$$\|\mathbf{A}\| \|\mathbf{x}^*\| \geq \|\mathbf{b}\|$$

which means

$$\frac{\|\delta \mathbf{b}\|}{\|\mathbf{A}\| \|\mathbf{x}^*\|} \leq \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|}.$$

and that's it.

Direct vs Iterative Solvers

In exact arithmetic

- Direct solvers return $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{b}$
  Example: Gaussian elimination
- Iterative solvers return a sequence $\{\mathbf{x}_n\}$ which converges to $\mathbf{x}^*$
  Example: stay tuned

Stationary Iterative Methods and the Banach Lemma

Fixed point (or Richardson or Picard) iteration for the linear problem

$$\mathbf{x} = \mathbf{M}\mathbf{x} + \mathbf{b}, \ (\text{so } \mathbf{A} = \mathbf{I} - \mathbf{M}) \ .$$

is

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n + \mathbf{b}.$$

$\mathbf{M}$ is called the **iteration matrix**.
Stationary means that the formula from $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$ does not depend on the history of the iteration.

The Banach Lemma: Convergence of Stationary Iterative Methods

Let $\mathbf{M} \in \mathbf{R}^{N \times N}$. Assume that

$$\|\mathbf{M}\| < 1$$

for some induced matrix norm. Then

- $(\mathbf{I} - \mathbf{M})$ is nonsingular
- $(\mathbf{I} - \mathbf{M})^{-1} = \sum_{l=0}^{\infty} \mathbf{M}^l$
- $\|(\mathbf{I} - \mathbf{M})^{-1}\| \leq (1 - \|\mathbf{M}\|)^{-1}$

Proof of Banach Lemma: I

We will show that the series

$$\sum_{l=0}^{\infty} \mathbf{M}^l = (I - \mathbf{M})^{-1}.$$

The partial sums

$$\mathbf{S}_k = \sum_{l=0}^{k} \mathbf{M}^l$$

form a Cauchy sequence in $R^{N \times N}$. To see this note that for all $m > k$

$$\|\mathbf{S}_k - \mathbf{S}_m\| \leq \sum_{l=k+1}^{m} \|\mathbf{M}^l\|.$$

And ...

Proof of Banach Lemma: II

$\|\mathbf{M}^l\| \le \|\mathbf{M}\|^l$ because $\| \cdot \|$ is a matrix norm that is induced by a vector norm. Hence

$$\|\mathbf{S}_k - \mathbf{S}_m\| \le \sum_{l=k+1}^{m} \|\mathbf{M}\|^l = \|\mathbf{M}\|^{k+1} \left(\frac{1 - \|\mathbf{M}\|^{m-k}}{1 - \|\mathbf{M}\|} \right) \to 0$$

as $m, k \to \infty$. So the series converges. Let

$$\mathbf{S} = \sum_{l=0}^{\infty} \mathbf{M}^l$$

Proof of Banach Lemma: III

Clearly

$$\mathbf{MS} = \sum_{l=0}^{\infty} \mathbf{M}^{l+1} = \sum_{l=1}^{\infty} \mathbf{M}^{l} = \mathbf{S} - \mathbf{I} \text{ and so}$$

$$(\mathbf{I} - \mathbf{M})\mathbf{S} = \mathbf{I} \text{ and } \mathbf{S} = (\mathbf{I} - \mathbf{M})^{-1}.$$

Finally

$$\|(\mathbf{I} - \mathbf{M})^{-1}\| \leq \sum_{l=0}^{\infty} \|\mathbf{M}\|^{l} = (1 - \|\mathbf{M}\|)^{-1}.$$

Convergence for Stationary Iterative Methods

If $\|\mathbf{M}\| < 1$ for any induced matrix norm then the stationary iteration

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n + \mathbf{b}$$

converges for all $\mathbf{b}$ and $\mathbf{x}_0$ to $\mathbf{x}^* = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{b}$

Proof: Clearly

$$\mathbf{x}_{n+1} = \sum_{l=0}^{n} \mathbf{M}^l \mathbf{b} + \mathbf{M}^n \mathbf{x}_0 \to (\mathbf{I} - \mathbf{M})^{-1}\mathbf{b} = \mathbf{x}^*.$$

Convergence Speed

Let $\|\mathbf{M}\| = \alpha < 1$ and $x^* = (\mathbf{I} - \mathbf{M})^{-1} c$. Then

$$\|\mathbf{x}^* - \mathbf{x}_n\| \leq \alpha^n \|\mathbf{x}^* - \mathbf{x}_0\|.$$

Proof:

$$\begin{aligned} \|\mathbf{x}^* - \mathbf{x}_n\| &= \| \textstyle\sum_{l=n}^{\infty} M^l c - \mathbf{M}^n \mathbf{x}_0 \| \\ &= \|\mathbf{M}(\textstyle\sum_{l=n-1}^{\infty} \mathbf{M}^l c - \mathbf{M}^{n-1} \mathbf{x}_0)\| = \|\mathbf{M}(\mathbf{x}^* - \mathbf{x}_{n-1})\| \\ &\leq \alpha \|\mathbf{x}^* - \mathbf{x}_{n-1}\| \leq \cdots \leq \alpha^n \|\mathbf{x}^* - \mathbf{x}_0\|. \end{aligned}$$

Spectral Radius

The spectrum of $\mathbf{M}$ $\sigma(\mathbf{M})$, is the set of eigenvalues of $\mathbf{M}$. The spectral radius of $\mathbf{M}$ is

$$\rho(\mathbf{M}) = \max_{\lambda \in \sigma(\mathbf{M})} |\lambda|$$

Theorem $\rho(\mathbf{M}) < 1$ if and only if $\|\mathbf{M}\| < 1$ for some induced matrix norm.

A stationary iterative method $\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n + \mathbf{b}$ converges for all initial iterates and right sides if and only if $\rho(\mathbf{M}) < 1$.

The spectral radius does not depend on any norm.

Predicting Convergence

Start with $\mathbf{x}_0 = 0$. Suppose you know that $\|\mathbf{M}\| \le \alpha < 1$. Then

$$\mathbf{e}_{n+1} = \mathbf{x}^* - \mathbf{x}_{n+1} = (\mathbf{M}\mathbf{x}^* + c) - (\mathbf{M}\mathbf{x}_n + c) = \mathbf{M}\mathbf{e}_n$$

Hence $\|\mathbf{e}_n\| \le \alpha^{n-1}\|\mathbf{e}_0\| = \alpha^{n-1}\|\mathbf{b}\|$ and

$$\|\mathbf{e}_n\| \le \tau\|\mathbf{b}\| \text{ if } \alpha^{n-1} < \tau$$

or $n > 1 + \log(\tau)/\log(\alpha)$.
You do not have to know what the right norm is to get convergence.

Preconditioned Richardson Iteration

If $\|\mathbf{I} - \mathbf{A}\| < 1$ then one can apply Richardson iteration directly to $\mathbf{Ax} = b$

$$\mathbf{x}_{n+1} = (\mathbf{I} - \mathbf{A})\mathbf{x}_n + \mathbf{b}$$

Sometimes one can find a approximate inverse $\mathbf{B}$ for which

$$\|\mathbf{I} - \mathbf{BA}\| < 1$$

and precondition with $\mathbf{B}$ to obtain

$$\mathbf{BAx} = \mathbf{Bb} \text{ and the iteration is } \mathbf{x}_{n+1} = (\mathbf{I} - \mathbf{BA})\mathbf{x}_n + \mathbf{Bb}$$

Preconditioning

Preconditioning $\mathbf{Ax} = \mathbf{b}$ means using a **preconditioner B** to make the system easier to solve.

- **Left preconditioning**: Solve $\mathbf{BAx} = \mathbf{Bb}$
- **Right preconditioning**: Solve $\mathbf{ABy} = \mathbf{B}$ and then set $\mathbf{x} = \mathbf{By}$.

Approximate Inverse Preconditioning: I

Theorem: If $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices and $\mathbf{B}$ is an approximate inverse of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ are both nonsingular and

$$\|\mathbf{A}^{-1}\| \leq \frac{\|\mathbf{B}\|}{1 - \|\mathbf{I} - \mathbf{BA}\|}, \quad \|\mathbf{B}^{-1}\| \leq \frac{\|\mathbf{A}\|}{1 - \|\mathbf{I} - \mathbf{BA}\|},$$

and

$$\|\mathbf{A}^{-1} - \mathbf{B}\| \leq \frac{\|\mathbf{B}\| \|\mathbf{I} - \mathbf{BA}\|}{1 - \|\mathbf{I} - \mathbf{BA}\|}, \quad \|\mathbf{A} - \mathbf{B}^{-1}\| \leq \frac{\|\mathbf{A}\| \|\mathbf{I} - \mathbf{BA}\|}{1 - \|\mathbf{I} - \mathbf{BA}\|}.$$

Approximate Inverse Preconditioning: II

Proof: Let $\mathbf{M} = \mathbf{I} - \mathbf{BA}$. The Banach Lemma implies that

$$\mathbf{I} - \mathbf{M} = \mathbf{I} - (\mathbf{I} - \mathbf{BA}) = \mathbf{BA}$$

is nonsingular. Hence both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular. Moreover

$$\|\mathbf{A}^{-1}\mathbf{B}^{-1}\| = \|(\mathbf{I} - \mathbf{M})^{-1}\| \leq \frac{1}{1 - \|\mathbf{M}\|} = \frac{1}{1 - \|\mathbf{I} - \mathbf{BA}\|}.$$

Approximate Inverse Preconditioning: III

Use $\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{B}$ to get the first part

$$\|\mathbf{A}^{-1}\| \leq \|\mathbf{B}\|\|(\mathbf{I} - \mathbf{M})^{-1}\| \leq \frac{\|\mathbf{B}\|}{1 - \|\mathbf{I} - \mathbf{BA}\|}.$$

The second pair of inequalities follows from

$$\mathbf{A}^{-1} - \mathbf{B} = (\mathbf{I} - \mathbf{BA})\mathbf{A}^{-1}, \mathbf{A} - \mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{BA})$$

and the first.

Corollary: The half-way theorem

Theorem: Let $\mathbf{A}$ be nonsingular and

$$\|\mathbf{I} - \mathbf{A}^{-1}\mathbf{B}\| \le 1/2$$

Then $\mathbf{B}$ is nonsingular and

$$\|\mathbf{B}^{-1}\| \le 2\|\mathbf{A}^{-1}\|.$$

proof: Let $\mathbf{M} = \mathbf{I} - \mathbf{A}^{-1}\mathbf{B}$ and plug into previous stuff.